Degenerations of $(1,7)$–polarized abelian surfaces

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0 Introduction

The moduli space $A(1,7)$ of $(1,7)$–polarized abelian surfaces with a level structure was shown by Manolache and Schreyer to be rational with compactification $V(\mathscr{K}_4)$ a Fano 3-fold V_{22} [13]. This compactification is obtained considering the embedding of the abelian surfaces with their level structure in $\mathbf{P}V_0$, where V_0 is the Schrödinger representation of the Heisenberg group of level 7. Any such embedding is invariant under the action of G_7, an extension of H_7 by an involution. The fixed points of this involution and its conjugates in G_7 form an H_7-orbit of planes $\mathbf{P}_2^+$ and 3-spaces $\mathbf{P}_3^-$. The 3-fold $V(\mathscr{K}_4)$ parametrizes, what we denote by generalized G_7-embedded abelian surfaces. Every such surface intersects $\mathbf{P}_2^+$ in a finite subscheme of length 6, which we may classify by its type, namely the length of its components.

In this paper we prove the

Theorem *Let A a generalized G_7–embedded abelian surface of $\mathbf{P}V_0$, then according to the type of $\zeta_A = A \cap \mathbf{P}_2^+$ the surface A is*

type of ζ_A	*description*
$(1,1,1,1,1,1)$	*smooth and abelian*
$(2,1,1,1,1)$	*translation scroll* $(E,\pm\sigma)$ *with* $2\cdot\sigma \neq 0$
$(3,1,1,1)$	*tangent scroll* $(E,0)$
$(2,2,2)$	*doubled translation scroll* $(E,\pm\sigma)$ *with* $2\cdot\sigma = 0$ *and* $\sigma \neq 0$
$(2,2,1,1)$	*union of seven quadrics*
$(4,2)$	*union of seven doubled projective planes*
$(2,2,2)_s$	*union of fourteen projective planes*

The two occurring $(2,2,2)$ cases (abusively denoted by $(2,2,2)$ and $(2,2,2)_s$) are special ones and are described in Fig. 1 (subsection 4.1).

In the first section we first recall some basic facts on $(1,7)$–polarized abelian surfaces with level structure, and construct a compactification of their moduli space which turns out to be the same as the one constructed by Manolache and Schreyer. This alternative proof of their result (note that the main argument appears in their paper) is inspired by the case of $(1,5)$ polarizations [2], in particular we do not use syzygies (although both constructions coincide in the $(1,5)$ case!).

The second section deals essentially with remarks on prime Fano threefolds of index 12. For the one which is endowed by a faithful action of $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$ we give a careful description of the boundary.

In the last section we prove the theorem by interpreting the boundary in terms of surfaces in $\mathbf{P}^6 = \mathbf{P}V_0$.

Note that A. Marini investigates also such degenerations in [14] from a different interpretation of $V(\mathscr{K}_4)$, namely that of twisted cubics apolar to the "Kleinian" net of quadrics.

Notations

The base field is the one of complex numbers $\mathbf{C}$. If R is a vector space, the Veronese map from R to S^nR (as well as its projectivisation) will be denoted by ν_n :

$$R \xrightarrow{\nu_n} S^nR\ .$$

If $s \in \mathrm{Hilb}(n, \mathbf{P}R)$ the type of s (*i.e.* the associated length partition of n) will be labelled λ_s :

$$s \xrightarrow{\lambda} \lambda_s .$$

If H is a hypersurface of $\mathbf{P}R$ then $e_H = 0$ is an equation of H.

The irreducible representations of $\mathrm{SL}(2,\mathbf{F}_7)$ will be denoted by $\mathbf{C}$, W_3, $W_3^\vee$, U_4, $U_4^\vee$, W_6, U_6, $U_6^\vee$, W_7, W_8 and U_8. The algebra of representations of the group $\mathrm{SL}(2,\mathbf{F}_7)$ is a quotient of

$$\mathbf{Z}[\mathbf{C}, W_3, W_3^\vee, U_4, U_4^\vee, W_6, U_6, U_6^\vee, W_7, W_8, U_8]$$

where $\mathbf{C}$ denotes the trivial representation, W_n denotes an irreducible $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$–module of dimension n and U_n denotes an irreducible $\mathrm{SL}(2,\mathbf{F}_7)$–module of dimension n on which $\mathrm{SL}(2,\mathbf{F}_7)$ acts *faithfully*.

The corresponding table of multiplication can be found in [13] and [5] with the following possible identifications

[5]	V_1	$V_3 = V_-$	V_3^*	$V_4 = V_+$	V_4^*	V_6	V_6'	$V_6'^*$	V_7	V_8'	V_8
[13]	I	W	W'	U	U'	T	T_1	T_2	L	M_1	M_2
$\times$	$\mathbf{C}$	W_3	$W_3^\vee$	U_4	$U_4^\vee$	W_6	U_6	$U_6^\vee$	W_7	U_8	W_8
	$\cdot$	$\mathbf{P}_2^+$	$\check{\mathbf{P}}_2^+$	$\mathbf{P}_3^-$	$\check{\mathbf{P}}_3^-$	$\mathbf{P}_5^+$	$\mathbf{P}_5^-$	$\check{\mathbf{P}}_5^-$	$\mathbf{P}_6^+$	$\mathbf{P}_7^-$	$\mathbf{P}_7^+$

Note that what are denoted by $\mathbf{P}_2^+$ and $\mathbf{P}_3^-$ are respectively denoted by $\mathbf{P}_-^2$ and $\mathbf{P}_+^3$ in [10].

The seven dimensional vector spaces $V_0, ..., V_5$ are irreducible H_7–modules, a useful multiplication table have been stolen (among many other things) in [13].

1 Moduli space : a compactification

Let A be an abelian surface, *i.e.* a projective complex torus $\mathbf{C}^2/\Lambda$ where Λ is a (maximal) lattice of $\mathbf{C}^2 \simeq \mathbf{R}^4$. Then the variety $Pic^0(A)$ turns to be an abelian surface as well (isomorphic to $(\mathbf{C}^2)^\vee/\Lambda^\vee$); this latter one is called the *dual* abelian surface of A and will be denoted by $A^\vee$. As additive group, the surface A acts on itself by translation, if $x \in A$ we will denote by τ_x the corresponding translation.

A line bundle of type $(1,7)$ on A is the data of an ample line bundle $\mathscr{L}$ such that the kernel of the isogeny

$$\varphi_{\mathscr{L}} : A \longrightarrow A^\vee, \qquad x \longrightarrow \tau_x^* \mathscr{L} \otimes \mathscr{L}^{-1}$$

is the affine plane (with origin) over $\mathbf{F}_7$, *i.e.* such that $ker(\varphi_{\mathscr{L}}) \simeq \mathbf{Z}_7 \times \mathbf{Z}_7$.

A $(1,7)$–polarization on A is an element of

$$\{(A, \varphi_{\mathscr{L}}) \mid \mathscr{L} \text{ is of type } (1,7)\}.$$

Thanks to Mumford, a coarse moduli space of $(1,7)$–polarized abelian surfaces exists, we will denote it by $M(1,7)$.

Now choose a *generic* abelian surface, say A, then $V_0 = \mathrm{H}^0(A, \mathscr{L})$ is of dimension 7. The group $ker(\varphi_{\mathscr{L}}) \simeq \mathbf{Z}_7 \times \mathbf{Z}_7$ becomes a subgroup of $\mathbf{P}\mathrm{SL}(V_0)$. It is certainly safer to work with linear representations rather than projective ones so we need to lift the action of $\mathbf{Z}_7 \times \mathbf{Z}_7$ on $\mathbf{P}V_0$ to an action of one of its central extensions on V_0. The Schur multiplier of $\mathbf{Z}_7 \times \mathbf{Z}_7$ is known to be μ_7 so any projective representation of $\mathbf{Z}_7 \times \mathbf{Z}_7$ is induced by a linear representation of what is called the "Heisenberg group of level 7" and denoted by H_7: that is to say for all $n \in \mathbf{N}^*$ and all projective representation ρ we get a cartesian diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] & \mathrm{SL}(n,\mathbf{C}) \arrow[r] & \mathbf{P}\mathrm{SL}(n,\mathbf{C}) \arrow[r] & 1 \\
1 \arrow[r] & \mu_7 \arrow[r] \arrow[u] & H_7 \arrow[r] \arrow[u] & \mathbf{Z}_7 \times \mathbf{Z}_7 \arrow[r] \arrow[u, "\rho"'] & 1
\end{tikzcd}$$

In this way V_0 becomes a H_7-module (of rank 7), this representation is called the "Schrödinger" representation of H_7. We have now a way to identified *all* the vector spaces $\mathrm{H}^0(A', \mathscr{L})$ for any abelian surface $A' \in M(1,7)$ as they are all isomorphic to V_0 as H_7-modules. This looks too good to be true. So what is wrong? We implicitly made an identification between $ker(\varphi_{\mathscr{L}})$ and $\mathbf{Z}_7 \times \mathbf{Z}_7$ and this is certainly defined up to $\mathrm{SL}(2, \mathbf{F}_7)$ only! So the construction is only invariant under $N_7 = H_7 \rtimes \mathrm{SL}(2, \mathbf{F}_7)$ which turns out to be the normaliser of H_7 in $\mathrm{SL}_7(\mathbf{C}) \simeq \mathrm{SL}(\mathbf{V_0})$ (this seems to be a collective agreement...).

So to any basis s of $ker(\varphi_{\mathscr{L}})$ corresponds an embedding

$$\zeta_s : A \longrightarrow \mathbf{P}V_0.$$

The group $\mathrm{SL}(2, \mathbf{F}_7)$ acts on the set of basis of $ker(\varphi_{\mathscr{L}})$ and we immediately get another complication (which will turn out to be quite nice after all):

$$\zeta_s(A) = \zeta_{-s}(A).$$

Let us denote by $G_7 = H_7 \rtimes \{-1, 1\} \subset N_7$. This group (after killing μ_7) is in general the full group of automorphisms of the surface $\zeta_s(A)$: if b is any element of $\mathbf{Z}_7 \times \mathbf{Z}_7$ and $\tau_b : \mathbf{P}V_0 \longrightarrow \mathbf{P}V_0$ is the involution induced by the corresponding "-1" of G_7, then τ_b leaves $\zeta_s(A)$ (globally) invariant and is induced by the "opposite" map $x \mapsto -x$ on A for a good choice of the image of the origin on $\zeta_s(A)$. In other words, $\tau_b \cdot \zeta_s = \zeta_{-s}$.

As the cardinality of $\mathrm{SL}(2, \mathbf{F}_7)/\{-1, 1\} = \mathbf{P}\mathrm{SL}(2, \mathbf{F}_7)$ is 168, each element of $M(1,7)$ will be embedded in $\mathbf{P}V_0$ in 168 ways (distinct in general). We get a brand new moduli space considering a $(1,7)$–polarized abelian surface together with one of its embedding, this moduli space will be denoted by $A(1,7)$:

$$A(1,7) = \{((A, \varphi_{\mathscr{L}}), s) |\, (A, \varphi_{\mathscr{L}}) \in M(1,7),\ s \text{ is a basis of } ker(\varphi_{\mathscr{L}})\}/\square$$

in which the equivalence relation $\square$ is the expected one, $(X_1, s_1) \,\square\, (X_2, s_2)$ if $\zeta_{s_1}(X_1) = \zeta_{s_2}(X_2)$ (fortunately, this implies $X_1 = X_2$). The choice of basis (or embedding) is the level structure refered to in the introduction.

Here are useful remarks:

i. The surface $\zeta_s(A)$ is of degree 14;

ii. by construction if $x \in A$ then the set of 49 points $\zeta_s(\varphi_{\mathscr{L}}^{-1}(\varphi_{\mathscr{L}}(x)))$ is an orbit under the action of H_7 (or H_7/μ_7 if we want to be precise);

iii. the above construction works as well for elliptic curves, so in particular $\mathbf{P}V_0$ contains naturally G_7–invariant embedded elliptic curves (of degree 7);

iv. if $b \in \mathbf{Z}_7 \times \mathbf{Z}_7$ the involution τ_b induces a $\mathrm{SL}(2, \mathbf{F}_7)$–module structure on V_0, as such a module V_0 splits in $V_0 = W_3 \oplus U_4$ where both W_3 and U_4 are irreducible $\mathrm{SL}(2, \mathbf{F}_7)$–modules of dimension 3 and 4 respectively (such that $S^3 W_3^\vee \simeq S^2 U_4$). The projective plane $\mathbf{P}W_3$ and the projective space $\mathbf{P}U_4$ in $\mathbf{P}V_0$ are point by point invariant by the involution τ_b. For a given $b \in \mathbf{Z}_7 \times \mathbf{Z}_7$ these two spaces will often be denoted by $\mathbf{P}_2^+$ and $\mathbf{P}_3^-$ (the signs come from the following: W_3 is also a $\mathbf{P}\mathrm{SL}(2, \mathbf{F}_7)$–module, *i.e.* $-1 \in \mathrm{SL}(2, \mathbf{F}_7)$ acts trivially on it, but $\mathrm{SL}(2, \mathbf{F}_7)$ acts faithfully on U_4);

v. if E is a G_7–invariant elliptic curve in $\mathbf{P}V_0$ then the curve E intersects any $\mathbf{P}_2^+$ in one point (corresponding to the image of 0) and any $\mathbf{P}_3^-$ in 3 points (corresponding to its non trivial 2–torsion points);

vi. the latter holds also for abelian surfaces, with decomposition $6 + 10$ corresponding to the odd and even 2–torsion points ([12]);

vii. by adding a finite set of G_7–invariant heptagons to the union of the G_7–invariant embedded elliptic curves of degree 7, one gets a birational model of the Shioda modular surface of level 7. It intersects each $\mathbf{P}_2^+$ in a plane quartic curve $\mathscr{K}'_4$, the socalled Klein quartic curve ([15] or [8] which contains original references to Klein).

Following what happens in the $(1,5)$ case we next consider the rational map

$$\kappa : \mathbf{P}V_0 \dashrightarrow \mathbf{P}(\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7})^\vee$$

i.e. the blowup of $\mathbf{P}V_0$ by the linear system of G_7–invariant septimics . *In what follows, by a 'G_7–invariant septimic ' we always (and imprudently) mean a septimic in this linear system.* The isomorphism of $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$–modules $\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7} \simeq W_7 \oplus \mathbf{C}$ shows that κ takes, *a priori*, its values in a $\mathbf{P}^7$. We will see that the image of κ is in fact contained in $\mathbf{P}W_7$. First we analyse the base locus of these septimic hypersurfaces.

Lemma 1.1 *A G_7–invariant septimic hypersurfaces of $\mathbf{P}V_0$ contains any of the forty–nine projective spaces $\mathbf{P}_3^-$.*

Proof. Obviously the vector space $\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7}$ is a $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$–module, so considering the restriction to any projective space $\mathbf{P}_3^- = \mathbf{P}U_4$ we get a map $\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathscr{O}_{\mathbf{P}U_4}(7)) = S^7U_4^\vee$ which need to be $\mathrm{SL}(2,\mathbf{F}_7)$–equivariant (the entire collection of $\mathbf{P}_3^-$'s being invariant under the action of G_7). But U_4 is a faithful module for $\mathrm{SL}(2,\mathbf{F}_7)$ and 7 is odd,so the map is the zero map. □

Using Bezout theorem we get

Corollary 1.2 *A G_7–invariant septimic hypersurface of $\mathbf{P}V_0$ contains any G_7–invariant elliptic curve of $\mathbf{P}V_0$ as well as its translation scroll by a non trivial 2–torsion point.*

Notice that our forty nine $\mathbf{P}_2^+$ constitute an orbit under G_7, so it makes sense to consider *the* surface $\kappa(\mathbf{P}_2^+)$. As any objects we are interested in intersects these planes, this will certainly be a good point to understand this surface.

Corollary 1.3 *'The' plane $\mathbf{P}_2^+$ is mapped by κ to a Veronese surface of degree nine in $\mathbf{P}W_7$.*

Proof. Let $P \in \mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7}$. From the preceding corollary the hypersurface $\{P=0\}$ contains the Shioda modular surface, and in particular its 7–torsion points sections. So the septimic hypersurface $\{P=0\}$ intersects any plane $\mathbf{P}_2^+$ along a reducible septimic curve, union of the Klein quartic curve (which is the curve of 7–torsion points contained in a plane $\mathbf{P}_2^+$) and a cubic curve. The irreducible $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$–modules decomposition of the vector space $S^3W_3^\vee \simeq W_7 \oplus W_3$ brings us to the proclaimed result, if one notices the equality $W_7 \simeq W_7^\vee$. □

Remark 1.4 This phenomenon holds also in the $(1,5)$ case where $\mathbf{P}_2^+$ is mapped to a (projected) Veronese surface of degree 25 in a Grassmannian $\mathrm{Gr}(1,\mathbf{P}_3) \subset \mathbf{P}_5$ known as the bisecants variety of a certain rational sextic curve in $\mathbf{P}_3$. The image of the blow-up of 'the' line $\mathbf{P}_1^-$ intersects the latter Grassmannian along the sextic complex of lines contained in a plane of a *dual* sextic of $\mathbf{P}_3$. In this case, any G_5–embedded $(1,5)$–polarized abelian surface is mapped to a ten–secant plane to $\kappa(\mathbf{P}_2^+)$ (which intersects the sextic complex along six lines). Although the same kind of results are expected in our situation, here is a difference between the two cases, namely

Remark 1.5 The vector space $\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7}$ is not given by determinants of (symmetric) Moore matrices ([10]).

Proof. Let us recall first what is a Moore matrix; we have a nice isomorphism of irreducible N_7–modules (defined up to homothety) $S^2V_4 = U_4 \otimes V_0$ which induces a map $U_4 \longrightarrow S^2V_4 \otimes V_0^\vee$. For a good choice of basis in $V_0^\vee$ we get a 7×7 matrix with coefficients in $V_0^\vee$ which is called a (symmetric) "Moore matrix". Now considering determinants (*i.e.* the map $S^2V_4 \xrightarrow{\text{“}S^2\Lambda^7\text{”}} \mathbf{C}$) we get a first map

$$S^7U_4 \longrightarrow S^7V_0^\vee = \mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7)),$$

which composed with the projection to the invariant part yields a map

$$S^7U_4 \longrightarrow \mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7}.$$

This latter one is certainly zero: the action of $-1 \in \mathrm{SL}(2, \mathbf{F}_7)$ cannot be trivial on any $\mathrm{SL}(2, \mathbf{F}_7)$–invariant subspace of the vector space S^7U_4. □

Nevertheless *anti-symmetric* Moore matrices play a fundamental role in the $(1, 7)$ case. They are defined by the isomorphism of irreducible N_7–modules $\Lambda^2V_4 = W_3^\vee \otimes V_0$. The locus (in $\mathbf{P}V_0$) where such matrix drops its rank are Calabi-Yau threefolds (see [10]) and will appear in subsection 3.3.

Proposition 1.6 *The image by the map* κ *of a* G_7*–embedded* $(1, 7)$*–polarized abelian surface is (generically) a projective plane and we have a factorization*

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A)$$

where the surface $A^\flat$ *is the blowup of the surface* A *along its intersection with the base locus of the* G_7*-invariant septimics .*

Proof. Assume the $(1, 7)$–polarization of $A \in M(1, 7)$ is given by a *very* ample line bundle, then from the G_7–equivariant resolution of the surface A in $\mathbf{P}V_0$ which can be found in [13], one can check (or read, it is easier *op. cit.*, appendix) that $\dim(\mathrm{H}^0(\mathscr{O}_A(7))^{G_7}) = 3$.

Supposing the map $\kappa|_A$ is finite (as it should be) then we have a factorization

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A) \ .$$

The first two maps (as well as their degree) come from the construction itself, the degree of the last one is due to Étienne Bezout.

If $\kappa|_A$ is not finite then it is composed with a pencil. We may assume that $\mathrm{Pic}(A)$ has rank 1, *i.e.* all curves are hypersurface sections or translates thereof. But no such curve is fixed by G_7, so $\kappa|_A$ has at most isolated base points. So the linear system is a subsystem of $|7 \cdot h|$. In particular if it composed with a pencil the fibers are curves in $|h|$ through the base locus. But there are clearly no such curve, since already the 49 $\mathbf{P}_3^-$ span $\mathbf{P}^6 = \mathbf{P}V_0$. Thus the map κ is finite on A and the proposition follows. □

Let us denote by $A(1, 7)^v$ the (open) subset of $A(1, 7)$ corresponding to $(1, 7)$–polarized abelian surfaces for which the polarization is given by a very ample line bundle. Using the preceding section we get a nice embedding $A(1, 7)^v$ in the (compact) variety of six secant planes to $\kappa(\mathbf{P}_2^+)$ and then a natural compactification $\overline{A(1, 7)^v}$ (see N. Manolache and F.-O. Schreyer's paper [13] for another proof) which comes with a geometric meaning; we just need to consider the proper transform of a six–secant plane of the Veronese surface by κ^{-1}. Moreover, *any* $(1, 7)$–polarized abelian surface is mapped into the hyperplane $\mathbf{P}W_7$ of $\mathbf{P}\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7}$ so their union is contained in a septimic hypersurface of $\mathbf{P}V_0$. So we have

Corollary 1.7 *The moduli space* $\overline{A(1, 7)^v}$ *is isomorphic to the unique prime Fano threefold of genus* 12 *which admits* $\mathbf{P}\mathrm{SL}(2, \mathbf{F}_7)$ *as its automorphisms group. The universal* $(1, 7)$*–polarized abelian surface with level* 7 *structure is birational to the unique* N_7*–invariant septimic hypersurface of* $\mathbf{P}V_0$*.*

Proof. Let us denote by X_7 the unique N_7–invariant septimic hypersurface of $\mathbf{P}V_0$ and by B_κ the base locus of the G_7–invariant septimic hypersurfaces. Put $Y_4 = \overline{\kappa(X_7 \backslash B_\kappa)} \subset \mathbf{P}W_7$ and consider the diagram

$$\begin{tikzcd} & I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_7\backslash B_\kappa \arrow[r, "\kappa"] & Y_4 & \mathbf{G}(3,\, W_7) \end{tikzcd}$$

where $I \subset \mathbf{P}W_7 \times \mathbf{G}(3, W_7)$ denotes the graph of the incidence correspondence between $\mathbf{P}W_7$ and the (projective) fibers of the tautological sheaf over $\mathbf{G}(3, W_7)$ and where p_1 and p_2 are the natural projections. In order to prove birationality we just need to prove that a general point of X_7 is contained in one (and only one) abelian surface. One first needs to remark, using representation theory for instance, that both the hypersurfaces X_7 and Y_4 are irreducible.

Let $A \in A(1,7)^{\nu}$ a G_7–embedded abelian surface. We have:

- the septimic X_7 contains the surface A;
- the surface A intersects $\mathbf{P}_2^+$ along a reduced scheme;
- the surface A *is not contained* in the base locus B_κ.

The only non obvious fact is the third item. But B_κ intersects $\mathbf{P}_2^+$ along a Klein quartic curve $\mathscr{K}'_4$ so if we had $A \subset B_\kappa$ this would imply the non emptiness of $A \cap \mathscr{K}'_4$ and in such cases $A \cap \mathbf{P}_2^+$ admits a double point (see e.g. section 2 below) contradicting the second item. Next the map $A \longmapsto A \cap \mathbf{P}_2^+$ is injective (see [13]) so the plane $\overline{\kappa(A \backslash B_\kappa)}$ entirely characterizes the surface A. Summing up all we know, we can pretend that two distincts surfaces A and A' intersect each other either on

- the threefold B_κ (which is of codimension 2 in X_7);
- the preimage by κ of the points in $Y_4 \subset \mathbf{P}W_7$ which are contained in more than one six-secant plane to the Veronese surface $\overline{\kappa(\mathbf{P}_2^+ \backslash \mathscr{K}'_4)}$.

Since A is not contained in B_κ, it remains to show that A is not contained in the second locus. But one proves easily that the second locus is 2-dimensional, being the preimage of the reunion of the Veronese surface $\overline{\kappa(\mathbf{P}_2^+ \backslash \mathscr{K}'_4)}$ itself and its ruled surface of trisecant lines (for which the base is isomorphic to the Klein quartic curve $\mathscr{K}_4$ of the dual plane $\check{\mathbf{P}}_2^+$). □

Remark 1.8 Notice that one can also prove (using Schubert calculus) that the hypersurface Y_4 has degree four in $\mathbf{P}W_7$ (this is true for any collection of six-secant planes to such projected Verenose surface).

2 Fano threefolds V_{22}

Let us recall a characterization of prime Fano threefolds of genus 12 (cf [16]).

Definition-Proposition 2.1 *Any Fano threefold of index 1 and genus 12 is isomorphic to the variety of sums of power* $VSP(F,6) = \overline{\{(\ell_1,...,\ell_6) \in \mathrm{Hilb}_6\, \mathbf{P}W^\vee \mid e_F \equiv \Sigma_{i=1}^6 e_{\ell_i}^4\}}$ *of a plane quartic curve F. Conversely, if F is not a Clebsch quartic (i.e. its catalecticant invariant vanishes), then $VSP(F,6)$ is a Fano threefold of index 1 and genus 12. Its anti-canonical model is denoted by V_{22}.*

2.1 Construction

Let W be an irreducible $\mathrm{SL}(3,\mathbf{C})$-module of dimension 3, we have a decomposition of $\mathrm{SL}(3,\mathbf{C})$-modules ([9])

$$S^2(S^2W)^\vee = S^4W^\vee \oplus S^2W$$

generating an exact sequence

$$0 \longrightarrow S^4W^\vee \longrightarrow \mathrm{Hom}(S^2W, S^2W^\vee).$$

So a plane quartic F in $\mathbf{P}W$ whose equation is given by 'an' element e_F of $S^4W^\vee$ gives rise to 'a' morphism $\alpha_F : S^2W \longrightarrow S^2W^\vee$ and a quadric $\mathbf{Q}_F$ in $\mathbf{P}(S^2W)$ of equation $\alpha_F(x)\cdot x = 0$ or even $x \cdot \alpha_F(x) = 0$ by the canonical identification $S^2W = (S^2W^\vee)^\vee$. From the equality $\mathrm{h}^0(\mathscr{I}_{\nu_2(F)}(2)) = 7$, we get a *characterization of this quadric* by the two properties:

i. the two forms on W defined by $\alpha_F(\nu_2(-)) \cdot \nu_2(-)$ and e_F are proportional *i.e.* the quadric $\mathbf{Q}_F$ and the Veronese surface $\nu_2(\mathbf{P}W)$ intersects along the image of the plane quartic F under ν_2;

ii. the quadric $\mathbf{Q}_F$ is apolar to the Veronese surface $\nu_2(\mathbf{P}W^\vee)$ of $\mathbf{P}S^2W^\vee$ *i.e.* apolar to each element of the vector space $\mathrm{H}^0(\mathscr{I}_{\nu_2(\mathbf{P}W^\vee)}(2)) \simeq S^2W^\vee \subset S^2(S^2W)$.

Lemma 2.2 (Sylvester) *The minimal integer n for which $VSP(F,n)$ is non empty is the rank of α_F (called the catalecticant invariant of the quartic curve).*

Proof. This well known result of Sylvester (see e.g. Dolgachev and Kanev [6], Elliot [7, page 294]) can be deduced from the following observation: let $n \in \mathbf{N}^*$, then

$$\nu_2(VSP(F,n)) = \{(p_1, ..., p_n) \in VSP(\mathbf{Q}_F, n) \mid p_\times \in \nu_2(\mathbf{P}W^\vee)\}.$$

Indeed if $\square = (\ell_1, ..., \ell_n) \in VSP(F,n)$ then $e_F \equiv \sum_{i=1}^{n} e_{\ell_i}^4$ for a good normalization of $e_{l_\times}$ and the quadric $\mathbf{Q} \subset \mathbf{P}S^2W$ of equation

$$e_{\mathbf{Q}} \equiv \sum_{i=1}^{n} e^2_{\nu_2(\ell_i)}$$

is endowed with the two properties which characterize the quadric $\mathbf{Q_F}$: the second one is a direct consequence of $\mathrm{H}^0(\mathscr{I}_{\nu_2(\mathbf{P}W^\vee)}(2)) \subset \mathrm{H}^0(\mathscr{I}_{\nu_2(\square)}(2))$ and the first one arises by construction. Applying ν_2^{-1} we get the required equality. $\square$

Define the vector space $Y_\ell \subset S^2W$ such that the line ℓ of the plane $\mathbf{P}W$ induces the exact sequence

$$0 \longrightarrow \mathbf{C} \cdot e_\ell^2 \longrightarrow S^2W^\vee \longrightarrow Y_\ell^\vee \longrightarrow 0\ ,$$

that is to say Y_ℓ is the orthogonal space (in S^2W) of e_ℓ^2.

Definition 2.3 *The subscheme $\mathscr{C}_\ell$ of the plane $\mathbf{P}W^\vee$ defined by $\mathscr{C}_\ell = \{x \in \mathbf{P}W^\vee \mid e_x^2 \in \alpha_F(Y_\ell)\} = \nu_2^{-1}(\alpha_F(Y_\ell))$ is called the anti–polar conic of the line ℓ (with respect to the quartic F).*

Alternatively, if α_F has maximal rank we have obviously $\mathscr{C}_\ell = \{x \in \mathbf{P}W^\vee \mid e_x^2 \cdot \alpha_F^{-1}(e_\ell^2) = 0\}$.
Set $n = \mathrm{rank}(\alpha_F)$; the construction of a point of $VSP(F,n)$ is now very easy by the following

Corollary 2.4 *A point $(\ell_1, ..., \ell_n)$ lies inside $VSP(F,n)$ if and only if $\ell_i \in \mathscr{C}_{\ell_j}$ when $i \neq j$,*

which is an easy consequence of the classical construction of a point of $VSP(\mathbf{Q}, n)$ when $\mathbf{Q}$ is a quadric of rank n.

We will need to understand

2.2 Conics on the anti-canonical model

Let V be the seven dimensional vector space defined by the exact sequence $0 \longrightarrow W \longrightarrow S^3W^\vee \xrightarrow{p_F} V \longrightarrow 0$ where the second map is induced by $F \in S^4W^\vee \subset \mathrm{Hom}(W, S^3W^\vee)$ and denote by $\mathscr{V}'_{2,9}$ the image of $\mathbf{P}W^\vee$ in $\mathbf{P}V$ by the Veronese embedding ν_3 composed with the third map p_F.

By definition $\square = (\ell_1, ..., \ell_n) \in VSP(F,6)$ if and only if the image by p_F of the six dimensional vector space (in $S^3W^\vee$) spanned by $e_{\ell_i}^3$ is of rank 3. Thus we get a map of $VSP(F,6)$ into the Grassmannian $G(3,V)$, by $(\ell_1, .., \ell_6) \mapsto p_F(< \ell_1, ..., \ell_6 >)$.

Remark 2.5 The image of $VSP(F,6)$ in the Plücker embedding of the Grassmannian is the anti-canonical model V_{22} of this Fano threefold, it is isomorphic to the variety of 6 secant planes to the projected Veronese surface $\mathscr{V}'_{2,9}$.

Now it is reasonable to talk about conics on V_{22}.

Denote by $F^\flat$ the dual quartic of $\mathbf{P}W$ of equation $\alpha_F^{-1}(e_\ell^2)\cdot e_\ell^2 = 0$, in other words we have $F^\flat = \{\ell \in \mathbf{P}W^\vee \,|\, \ell \in \mathscr{C}_\ell\}$ (the quartic $F^\flat$ reduces to a doubled conic when $n=5$) and by H_F the sextic of $\mathbf{P}W^\vee$ given by $H_F = \{\ell \in \mathbf{P}W^\vee \,|\, \mathrm{rank}(\alpha_F^{-1}(e_\ell^2)) \leqslant 1\}$. Let $\ell \in \mathbf{P}W^\vee \setminus H_F$, then the anti–polar conic $\mathscr{C}_\ell$ is smooth and we can write the abstract rational curve $\overline{\mathscr{C}_\ell}$ as $\mathbf{P}^1 = \mathbf{P}S_1$ with $\dim S_1 = 2$, *i.e.* we put $S_1 = \mathrm{H}^0(\mathscr{O}_{\overline{\mathscr{C}}_\ell}(1))$. Put $S_n := S^n S_1$, then $\mathbf{P}S_n$ is identified with the divisors of degree n on $\mathbf{P}S_1$ and we have

Lemma 2.6 *The set of divisors of degree 5 on the anti–polar conic $\mathscr{C}_\ell$ given by $\{\ell' + \mathscr{C}_\ell \cap \mathscr{C}_{\ell'},\ \ell' \in \mathscr{C}_\ell\}$ is a projective line in $\mathbf{P}S_5$. This g_1^5 admits a base point if $\ell \in F^\flat$.*

Proof. Let D be such a divisor. By corollary **2.4** the divisor D is completely determined by *any one* of its (sub)–divisor of degree 1, so the variety of such divisors is a curve of first degree in $\mathbf{P}S_5$. $\square$

Corollary 2.7 *The points of $VSP(F,6)$ which contain a given line ℓ describe a conic C_ℓ on the anti-canonical model V_{22}. The two conics C_ℓ and $\mathscr{C}_\ell$ have the same rank.*

Proof. Let ℓ outside H_F. Then the image of $\mathscr{C}_\ell$ by ν_3 is a rational normal sextic projected by the map p_F to a smooth sextic inside $\mathbf{P}^4_\ell := \mathbf{P}(p_F(\mathrm{H}^0(\mathscr{O}_{\mathscr{C}_\ell}(6))))$ — we have an injection $\mathrm{H}^0(\mathscr{O}_{\mathscr{C}_\ell}(6)) \subset S^3W^\vee$ and it is a simple matter to check $\mathrm{H}^0(\mathscr{O}_{\mathscr{C}_\ell}(6)) \cap \ker(p_F)$ is of dimension 2, moreover identifying $\ker(p_F)$ and W we have $\mathbf{P}(\mathrm{H}^0(\mathscr{O}_{(\mathscr{C}_\ell)}(6)) \cap W) = \ell \subset \mathbf{P}W$ —. Now $\mathbf{P}^4_\ell$ also contains the image of $\nu_3(\ell)$ and projecting from this latter point the sextic becomes :

i. a rational sextic on a quadric of a $\mathbf{P}^3$ generically;

ii. a rational quintic on a quadric of a $\mathbf{P}^3$ if $\ell \in F^\flat$.

These curves are obviously on a quadric, since a six secant plane to $\mathscr{V}'_{2,9}$ passing through the point $p_F(\nu_3(\ell))$ will be mapped to a 5-secant (*resp* 4-secant) line to this rational curve.

Now if $\ell \in H_F$, $\mathscr{C}_\ell$ breaks in two lines, say ℓ_1 and ℓ_2. We get two systems of six secant planes to $\mathscr{V}'_{2,9}$ containing $p_F(\nu_3(\ell))$, one of these intersects $p_F(\nu_3(\ell_1))$ in two *fixed* points and intersects the twisted cubic $p_F(\nu_3(\ell_2))$ along a pencil of divisors of degree 3. In particular, such collection is mapped to a line by κ. $\square$

Corollary 2.8 *If a is not on H_F the threefold $p_1 p_2^{-1}(C_a)$ is a quadric cone Γ_a of $\mathbf{P}^4_a$. If $a \in H_F$ the cone Γ_a splits in two $\mathbf{P}_3$'s.*

Remark 2.9 We already get a first interpretation in terms of abelian surfaces. Putting $W = W_3^\vee$ and choosing the unique $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$-invariant quartic $\mathscr{K}_4$ of $\mathbf{P}W = \check{\mathbf{P}}_2^+$ for F we get $V = W_7$, $\mathscr{V}'_{2,9} = \kappa(\mathbf{P}_2^+)$, $F^\flat = \mathscr{K}'_4$. Now if $a \in \mathbf{P}_2^+$ the proper transform of the quadric cone Γ_a by κ^{-1} is by [10] birational to a Calabi Yau threefold, and by the preceding corollary contains — when $\mathscr{C}_a$ is smooth — *two* dictinct pencils of special surfaces : the one induced by the six secant planes, parametrized by C_a and corresponding generically to abelian surfaces, and another one induced by the second ruling (parametrized by $\mathscr{C}_a$) of planes of the cone. We think that these last ones are the same as the ones evoked in [10, remark 5.7].

2.3 Boundary for the Klein quartic

Note that the boundary for the general case is easily deducible from what follows, but one need to introduce a covariant of F, this would be beyond the subject of this paper.

So in this section, we focus on the surface $\Delta_F = \{\square \in VSP(F,6) \,|\, \lambda_\square \neq (1^6)\}$ when the quartic F admits $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$ as its group of automorphisms.

We start by choosing a faithful embedding $1 \longrightarrow \mathrm{SL}(2,\mathbf{F}_7) \longrightarrow \mathrm{SL}(3,\mathbf{C})$, so the vector space W of our preceding section becomes a $\mathrm{SL}(2,\mathbf{F}_7)$–module (necessarily irreducible), say $W \simeq W_3^\vee$ and the decomposition $S^4W_3 = \mathbf{C} \oplus W_6 \oplus W_8$ allows us to consider the *unique* $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$–invariant quartic $\mathscr{K}_4$ of $\check{\mathbf{P}}_2^+ = \mathbf{P}W_3$. Such a

quartic is called a Klein quartic and becomes the quartic F of our preceding section. All the quartic covariants of F are equal to F (when non zero) and the Klein quartic $\mathscr{K}'_4 \subset \mathbf{P}_2^+$ is (by unicity) the quartic $F^\flat$ of the last section.

We'll need the classical

Lemma 2.10 *There is a unique* $\mathrm{SL}(2,\mathbf{F}_7)$*–invariant even theta characteristic* ϑ *on the genus* 3 *curve* $\mathscr{K}_4$ *(resp.* $\mathscr{K}'_4$*).*

Proof. The existence follows directly by the existence of a $\mathrm{SL}(2,\mathbf{F}_7)$–invariant injection $W_3 \longrightarrow S^2U_4$ so that one can illustrate $\mathscr{K}_4$ as the jacobian of a net of quadrics (in $\mathbf{P}U_4^\vee$). It is well known that such jacobian is endowed with an even theta characteristic (cf [1]). Reciprocally, such a theta characteristic on a curve of genus 3 comes with a net of quadrics and $\mathrm{Hom}_{\mathrm{SL}(2,\mathbf{F}_7)}(W_3, S^2U) \neq 0$ if and only if the four dimensional vector space U equals $U_4^\vee$ as $\mathrm{SL}(2,\mathbf{F}_7)$–module. □

We have

Proposition 2.11 *Let* $a \in \mathscr{K}'_4$ *and* $(x_1,x_2,x_3) \in \mathscr{K}'_4 \times \mathscr{K}'_4 \times \mathscr{K}'_4$ *such that* $\mathrm{h}^0(\vartheta + a - x_i) = 1$*, then the antipolar conic* $\mathscr{C}_{x_i}$ *of* x_i *with respect to* $\mathscr{K}_4$ *contains* x_j*.*

Proof. Let us leave the plane $\mathbf{P}_2^+$ and take a look at the configuration in $\mathbf{P}_5^+ = \mathbf{P}S^2W_3 = \mathbf{P}W_6 = \check{\mathbf{P}}_5^+$. The image of x_i by the Veronese embedding ν_2 lie on the quadric $Q_{\mathscr{K}'_4}$. On the other hand, noticing that $\mathrm{Hom}_{\mathrm{SL}_2\mathbf{F}_7}(\mathbf{C}, S^2S^2W_3) = \mathbf{C}$ this quadric can be interpreted

- as the inverse of the quadric $Q_{\mathscr{K}_4}$;
- as the Plücker embedding of the Grassmannian of lines of $\mathbf{P}_3^-$ using the $\mathrm{SL}(2,\mathbf{F}_7)$-invariant identification $S^2W_3 \simeq \Lambda^2U_4$.

Let us denote by $\mathscr{K}'_6$ the jacobian of the net of quadrics given by $W_3 \longrightarrow S^2U_4$ and remember that this curve is (by unicity) canonically isomorphic to $\mathscr{K}'_4$ itself. So $\nu_2(x_i)$ is a line in $\mathbf{P}_3^-$ (still denoted by $\nu_2(x_i)$) and this one turns out to be a trisecant line to the sextic $\mathscr{K}_6$ containing the image of a by the identification $\mathscr{K}'_4 = \mathscr{K}'_6$. This interpretation of the $(3,3)$ correspondence on $\mathscr{K}'_4 = \mathscr{K}'_6$ induced by the even theta characteristic as the incidence correspondence between $\mathscr{K}'_6$ and its trisecant lines is due to Clebsch. Now the three lines $\nu_2(x_i)$ are concurrent in a and then the three points $\nu_2(x_i)$ of $\mathbf{P}_5^+$ span a projective plane contained in the inverse of the quadric $Q_{\mathscr{K}_4}$. In particular, $\alpha^{-1}_{\mathscr{K}_4}(\nu_2(x_i)) \cdot \nu_2(x_j) = 0$ which is precisely what we need to claim that $x_j \in \mathscr{C}_{x_i}$. □

Notice that using the same geometric interpretation we get immediately

Corollary 2.12 *If* $a \in \mathscr{K}'_4$ *then the antipolar conic* $\mathscr{C}_a$ *intersects the hessian triangle* T_a *(ie the hessian of the polar cubic of* a *with respect to* $\mathscr{K}'_4$*) in points of the quartic* $\mathscr{K}'_4$ *(and* $\mathscr{C}_a \cap \mathscr{K}'_4 - 2a = T_a \cap \mathscr{K}'_4 - 2x_1 - 2x_2 - 2x_3$*)).*

Proposition 2.13 *Let* $p \in \Delta := \Delta_{\mathscr{K}'_4}$*, then there exists at least one point* a *in the support of* ζ_p *such that* $a \in \mathscr{K}'_4$ *and the type of* ζ_p *is one of the following*

	$a \notin \mathscr{H}_6$	$a \in \mathscr{H}_6$
	$2,1,1,1,1$	$2,2,1,1$
type of ζ_p	$3,1,1,1$	$4,2$
	$2,2,2$	$(2,2,2)_s$

Proof. From the preceding section, a point p of $VSP(\mathscr{K}_4, 6)$ is in Δ if and only if the support of ζ_p intersects the quartic curve $\mathscr{K}'_4$. So let $a \in \mathscr{K}'_4$, by corollary **2.4** the only thing to understand is the type of ζ_p when the point p moves along the conic $\mathcal{C}_a$. We have the alternative: the conic $\mathscr{C}_a$ is smooth (case *i*) or $a \in \mathscr{H}_6 := H_{\mathscr{K}'_4}$ (case *ii*).

i. Denote (once again) by S_n the two $(n+1)$-dimensional vector space $\mathrm{H}^0(\mathscr{O}_{\mathcal{C}_a}(n))$, we have $S_n = S^nS_1$. As $a \in \mathscr{K}'_4$, the $(1,5)$ correspondence between the two (isomorphic) rational curves $\mathcal{C}_a$ and $\mathscr{C}_a$ has a base point, namely the point a itself on $\mathscr{C}_a$ and then reduces to a $(1,4)$ correspondence. The induced pencil of divisors of degree 4 in $\mathbf{P}S_4$ intersects the variety of non reduced divisors in six points (as any generic pencil in $\mathbf{P}S_4$) and the expected types of ζ_p are hence $(2,1,1,1,1)$ generically, $(3,1,1,1)$ once and $(2,2,1,1)$ six times (each corresponding to a point of $\mathscr{C}_a \cap \mathscr{K}'_4 - \{a\}$). But by the preceding proposition, if $a' \in \mathscr{C}_a \cap \mathscr{K}'_4$ and $a \neq a'$ then the two conics $\mathscr{C}_a$ and $\mathscr{C}_{a'}$ intersect in $a + a' + 2a''$ with $a'' \in \mathscr{K}'_4$ hence the six expected $(2,2,1,1)$ on $\mathcal{C}_a$ become three $(2,2,2)$ for the particular Klein quartic. Notice that in such a case, the scheme ζ_p has a length decomposition $2\cdot(a+a'+a'')$ and there exists a point $b \in \mathscr{K}'_4$ so that $\mathrm{h}^0(\vartheta + b - x) = 1$ whenever $x \in \{a, a', a''\}$. Let us denote by b_x the intersection of $\mathscr{C}_x$ with the line $\overline{xb}$, then

$$a^3 \cdot b_a + a'^3 \cdot b_{a'} + a''^3 \cdot b_{a''} = 0$$

is an equation of $\mathscr{K}_4$.

ii. suppose now the point a is one of 24 points of intersection of the quartic $\mathscr{K}'_4$ and its Hessian $\mathscr{H}_6$. Such points come 3 by 3 and the group μ_3 acts on each triplet (so there is an order a_1, a_2, a_3 on such triplet). Put $a = a_1$. The conic $\mathscr{C}_{a_1}$ is no longer smooth and decomposes in two lines, say $\ell = a_1a_2$ and $\ell' = a_2a_3$. Each generic point b of the line ℓ gives us a point $2a_1 + b + b' + 2a_3 = \mathscr{C}_{a_1} \cap \mathscr{C}_b + a_1 + b$ of Δ (hence of type $(2,2,1,1)$) with $b' \in \ell$ defined such that the degree 4 divisor $a_2 + a_1 + b + b'$ on the line ℓ is harmonic. One can even provide the corresponding equation of the quartic $\mathscr{K}_4$

$$\varepsilon(\beta x + \alpha z)^4 - \varepsilon(\beta x - \alpha z)^4 - 2\alpha\beta\{(x + \varepsilon(\beta^2 z - \alpha^2 y))^4 - x^4\} + 2\alpha^3\beta((y + \varepsilon z)^4 - y^4) = 0$$

with coefficients in $\mathbf{C}[\varepsilon]/\varepsilon^2$. We can forget points of Δ arising from a point of ℓ', for such points can be constructed as the preceding ones by starting with the point a_2 instead of the point a_1. The possible degeneracies follow easily: when $(\alpha : \beta)$ tends to $(1:0)$ we get back to the well known $(2,2,2)_s$ case, the last equation becomes

$$(z + \varepsilon x)^4 - z^4 + (x + \varepsilon y)^4 - x^4 + (y + \varepsilon z)^4 - y^4 = 4\varepsilon\,(z^3x + x^3y + y^3z) = 0.$$

The last possible degeneration arises when $(\alpha : \beta)$ tends to $(1:0)$ in which case we get $(4,2)$ as partition of 6. □

3 Degenerated abelian surfaces

Now that we understood what was the boundary Δ of V_{22} in $\mathrm{Hilb}(6, \mathbf{P}_2^+)$ we try to understand it in terms of what we call 'degenerated abelian surfaces'. The method we are going to employ is very naïve : given $s \in \Delta$, find a surface A_s in $\mathbf{P}^6$ which intersects $\mathbf{P}_2^+$ along s and check that this surface is sent to a projective plane by the map κ, *ie* that $\mathrm{h}^0(\mathscr{I}_{A_s}(7))^{G_7} = 5$. Obviously translation scrolls are candidates and we'll see these are the good ones. In this section, we work up to the action of $\mathbf{P}\mathrm{SL}(2, \mathbf{F}_7)$.

3.1 Translation scrolls

We'll need the

Proposition 3.1 *Every translation scroll of an elliptic normal curve of degree 7 by a 2-torsion point is a smooth elliptic scroll of degree 7 and contains 3 elliptic normal curves of degree 7.*

Proof. cf. [4, proposition 1.1] or [15]. □

Let us start with $s \in \Delta_\lambda$ with $\lambda \in \{(2,1,1,1),(3,1,1,1)\}$. Only one point of s has a multiple structure, say $p \in \mathscr{K}'_4$ and the support of s consists in four *distinct* points on the conic $\mathscr{C}_p$. The stabilizer of $(s)_{\text{red}}$ under $\mathbf{P}\mathrm{SL}(2,\mathbf{C}) \simeq \mathrm{Aut}(\mathscr{C}_p)$ is in general isomorphic to $\mathbf{Z}_2^2$ (if it is bigger consider the subgroup $\{\mathrm{Id},(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$ in $\mathrm{Stab}_{\mathbf{P}\mathrm{SL}(2,\mathbf{C})}((s)_{\text{red}}) \subset \mathfrak{S}_4$). Consider the double cover E_s of $\mathscr{C}_p$ ramified at the four points $\mathbf{Z}_2^2 \cdot p$. It is a smooth elliptic curve and we choose p as origin on E_s. Note that, by proposition **2.11**, E_s depends only of p so it will be denoted by E_p. Now the linear system $|7 \cdot p|$ is a H_7-module and we can embed E_p in $\mathbf{P}^6$ in 168 distinct ways, one of them send $p \in E_p$ to $p \in \mathbf{P}_2^+$. Next $p \in s$ is doubled along a line that will intersect $\mathscr{C}_p$ in a further point, say p_s (of course $p = p_s$ if $\lambda = (3,1,1,1)$). Denote by σ_s one of the inverse images of p_s by the $2:1$ map $E_p \longrightarrow \mathscr{C}_p$.

Proposition 3.2 *The translation scroll (E_p,σ_s) intersects $\mathbf{P}_2^+$ along s and is mapped by κ to a plane.*

Proof. First the bisecant variety of E_p intersects $\mathbf{P}_2^+$ along $\mathscr{C}_p$: indeed such variety intersects $\mathbf{P}_2^+$ along a conic (E_p being of degree 7 and invariant under a symmetry which preserves $\mathbf{P}_2^+$). Next by the previous proposition such conic must contains the three pairs of points of ${\mathscr{K}'_4}^2$ such as (q_1,q_2) where $\{p,q_1,q_2\}$ are associated to a same point of $\mathscr{K}'_4$ under the $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$-invariant $(3,3)$ correspondence on $\mathscr{K}'_4$. By proposition **2.11**, this conic is nothing but $\mathscr{C}_p$.

Next as $\pm\sigma$ moves along $\mathscr{C}_p$, the set $(E_p,\pm\sigma)\cap\mathscr{C}_p - 2p$ describes a *pencil* of degree 4 divisors on $\mathscr{C}_p$, we need to identify this pencil with our pencil of degree 4 divisors on $\mathscr{C}_p$ (proof of proposition **2.13**, item *i*). But both pencils contain the three divisors of type $(2,2)$ such as $2q_1+2q_2$ so they are equal.

The point now is to prove that (E_p,σ_s) is mapped to a plane under κ *i.e.* that $\mathrm{h}^0(\mathscr{O}_{(E_p,\sigma_s)}(7))^{G_7} = 3$ or rather $\mathrm{h}^0(\mathscr{O}_{(E_p,\sigma_s)}(7))^{G_7} \leqslant 3$ by the previous paragraphs. The scroll (E_p,σ_s) is a $\mathbf{P}^1$–bundle over E_p (in two ways, these correspond to the choices σ_s and $-\sigma_s$ to define the scroll) so we have a map

$$(E_p,\sigma_s) \longrightarrow E_p$$

and if R denotes a generic fiber we get a sequence

$$\mathrm{H}^0(\mathscr{O}_{(E_p,\sigma_s)}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathscr{O}_R(7))^{G_7} \longrightarrow 0$$

which turns out to be exact: if a G_7–invariant septimic S contains the line R, then its intersection with (E_p,σ_s) contains $E_p \cup G_7 \cdot R$ which is of degree $7+49\times 2\times 1$. Now our scroll is of degree 14 and by Bezout we conclude $(E_p,\sigma_s) \subset S$. Using a semi-continuity argument we just need to find one fiber R such that $\mathrm{h}^0(\mathscr{O}_R(7))^{G_7} \leq 3$. Choose one of the forty-nine $\mathbf{P}_2^+$ and pick up one of the four lines of (E_p,σ_s) which intersects it. Then the restriction

$$\mathrm{H}^0(\mathscr{O}_{\mathbf{P}V_0}(7))^{G_7} \longrightarrow \mathrm{H}^0(\mathscr{O}_R(7))^{G_7}$$

is of rank 3 at most which is precisely what we needed. □

Of course to get the $(2,2,2)$ cases it is natural to make σ_s tend to a 2-torsion point. Then the translation scroll (E_p,σ_s) tends to a *smooth* scroll of degree 7 and everything is lost in virtue of the

Remark 3.3 Any translation scroll (E_p,σ_s) where σ_s is a non-trivial 2-torsion point of E_p is contained in all our G_7-invariant septimic hypersurfaces. Indeed such a scroll intersects any of the forty-nine $\mathbf{P}_3^-$'s along a line and contains the curve E_p itself, once again Bezout together with the inequality $49\times 1+7 > 7\times 7$ allow us to conclude. However we have

Proposition 3.4 *If $\lambda_s \in (2,2,2)$ then there exist an elliptic curve E_p, a two torsion point σ_s on E_p and a double structure on the translation scroll (E_p,σ_s) intersecting $\mathbf{P}_2^+$ along s and mapped to a plane by κ.*

Proof. Let $s \in \Delta_{(2,2,2)}$. By proposition **3.1** one and only one smooth translation scroll X intersects $\mathbf{P}_2^+$ along $(s)_{\text{red}}$ so we just need to find a double structure $\tilde{X}$ on X such that $\mathrm{h}^0(\mathscr{I}_{\tilde{X}}(7))^{G_7} = 5$. Now X contains two (in fact three by **3.1**) elliptic curves E_p and $E_{p'}$ and by definition is contained in the two corresponding bisecant

varieties S_{E_p} and $S_{E_{p'}}$. But these two varieties are the proper transforms by κ^{-1} of the two quadric cones Γ_p and $\Gamma_{p'}$ (cf corollary **2.8**). These two cones intersect along the six secant plane to $\kappa(\mathbf{P}_2^+)$ corresponding to s so we are done! The double structure is then easy to understand: one considers the double structures on $X\backslash E_p$ (*resp.* $X\backslash E_{p'}$) defined by the embedding $X \longrightarrow S_{E_p}$ (*resp.* $X \longrightarrow S_{E_{p'}}$) and such structures coincide on $X\backslash(E_p \cup E_{p'})$. $\square$

3.2 Union of seven quadrics

We still have to consider the missing cases, namely schemes s such that $\lambda_s \in \{(2,2,1,1),(2,2,2)_s,(4,2)\}$. These are degenerations of the preceding ones. A degenerated elliptic curve is nothing but a heptagon, and such curves come by triplets (E_0,E_1,E_2) (fig. 3) with $E_i = \bigcup_{k=0}^{6} \overline{e_k e_{k+1+i}}$ and $\{e_x\}_{x\in\mathbf{Z}_7}$ is an orbit of minimal cardinality under the action of G_7.

The Heisenberg action on each curve E_i reduces to an action of $\mathbf{Z}_7$. Let us denote by $\mathbf{P}_I$ the projective space spanned by the points $\{e_i\}_{i\in I}$.

For $i \in \{0,1,2\}$ put $B_i = \bigcup_{k=0}^{6} \mathbf{P}_{I_i^k}$ with $I_i^k = \{k+i+1, k-i-1, k+3i+3, k-3i-3\}$. Then the bisecant variety of E_i is $B_j + B_k$ with $\{i,j,k\} = \{0,1,2\}$.

Finally let us choose one of the forty nine $\mathbf{P}_2^+$ and suppose E_i intersects it in a_{i+1}. We are then ready to check the remaining cases:

Let $s \in \Delta$ such that $\lambda_s = (2,2,1,1)$ and $s = 2\cdot a_1 + 2\cdot a_3 + b + b'$ with b and b' on the line $\overline{a_1 a_2}$ (Fig. 1). So we have $s \in \mathcal{C}_{a_1} \cap \mathcal{C}_{a_3}$. The corresponding degenerated abelian surface A_s needs to be on the bisecant varieties of E_0 and E_2 so we have $A_s \subset B_1$. As B_1 is the union of seven $\mathbf{P}^3$'s the surface A_s is the union of seven quadrics.

When s moves along $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_3}$ we get the two other kinds of degenerations :

- if $\lambda_s = (2,2,2)_s$, then the surface A_s degenerates in the union of the fourteen planes $B_0 \cap B_1 \cup B_1 \cap B_2 \cup B_2 \cap B_0$;
- if $\lambda_s = (4,2)$, then the surface A_s degenerates in the union of the seven planes $\bigcup_{k=0}^{6} \mathbf{P}_{\{k,k+3,k-3\}}$ doubled along B_1.

3.3 The smooth case

To begin with let us

Remark 3.5 Choose coordinates $(y_i)_{i\in\{0,\dots,6\}}$ in V_0 together with one of the forty-nine $\mathbf{P}_3^-$'s of equations $y_4 = y_3,\ y_5 = y_2,\ y_6 = y_1$. Using the N_7-invariant isomorphism $\Lambda^2 V_4 = W_3^\vee \otimes V_0$ let us introduce for $x = (x_1 : x_2 : x_3) \in \mathbf{P}W_3$ and $y = (y_0 : - : y_4) \in \mathbf{P}_3^- \subset \mathbf{P}V_0$ the matrix

$$M_y(x) = \begin{pmatrix} x_2y_2 & -x_3y_1 - x_1y_3 & x_3y_0 & -x_2y_1 - x_1y_2 \\ -x_3y_3 & -x_3y_2 + x_2y_3 & -x_2y_1 + x_1y_2 & x_1y_0 \\ -x_1y_1 & x_2y_0 & x_3y_1 & x_3y_2 + x_2y_3 \end{pmatrix}.$$

It is the restriction to $\mathbf{P}_3^-$ of a skew-symmetric Moore matrix $M(x,y)$ (see remarks **1.5**, **2.9** and [10]). This matrix $M(x,y)$ defines the Calabi Yau threefold which is the strict transform of the cone Γ_x by κ^{-1}. For general $y \in \mathbf{P}_3^-$, $M_y(x)$ defines 6 points in $\mathbf{P}W_3$, meaning there is *one* abelian surface containing y and six Calabi Yau threefolds of the preceding type which contain this surface. But $M_y(x)$ may degenerate for special points $y \in \mathbf{P}_3^-$ (for such cases we get more than six points in $\mathbf{P}W_3$):

rank of $\Lambda^3 M_y(x)$	y in	x in	abelian varieties passing through y
3	$\mathscr{K}_6$	$\mathscr{H}_6$	have a 3–secant line
2	C_{18}	$\mathscr{K}_4'$	translation scrolls
1	Z	$\mathscr{K}_4' \cap \mathscr{H}_6$	reducible

where $\mathscr{K}_6$ is the unique $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$-invariant curve of degree 6 and genus 3 in $\mathbf{P}_3^-$, C_{18} is a $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$-invariant curve of degree 18 and genus 35 in $\mathbf{P}_3^-$ (analogue of the Bring curve in the $(1,5)$ case) and Z is the minimal orbit (of cardinality eight) under the action of $\mathbf{P}\mathrm{SL}(2,\mathbf{F}_7)$ on $\mathbf{P}_3^-$.

So far, and summing up all the results of this section, we need, in order to complete the proof of the theorem, to show that A is smooth and abelian provided the type of ζ_A is $(1,1,1,1,1,1)$. Now by the Enriques-Kodaira classification of surfaces (see [3, chapter VI]) complex tori are entirely characterized by their numerical invariants. So any generalized G_7–embedded abelian surface is an abelian surface provided it is smooth. We begin by the

Lemma 3.6 *A generalized G_7-embedded abelian surface A, singular along a curve, intersects 'the' plane $\mathbf{P}_2^+$ in a non reduced scheme.*

Proof. Let A a singular generalized G_7-embedded abelian surface. The proposition is obviously true if A is singular in codimension 0 that is to say if A carries a double structure. For such surface, the intersection of its reduced structure (of degree 7) with any $\mathbf{P}_2^+$ cannot be six distinct points so $\zeta_A = A \cap \mathbf{P}_2^+$, which is of length six, cannot be reduced.

By assumption the singular locus of A contains a curve C. We can also assume C is G_7-invariant (if not we replace C by its orbit under G_7).

If C has degree 7, it is necessarily elliptic and, being G_7 embedded, intersects $\mathbf{P}_2^+$ in a point of the Klein curve $\mathscr{K}'_4$ so we are done. Indeed the rationality of C (if irreducible) is totally excluded (such curve admits either a unique 4-secant plane, a unique trisecant line or a (unique) double point, this would be a contradiction with the irreducibility of V_0 as H_7–module), but C can still splits in the union of seven lines. We want to prove that C is a heptagon (that is to say elliptic). The stabilizer of one of the lines under the action of H_7 is isomorphic to $\mathbf{Z}_7$ so we get, on each line $\ell \subset C$, two fixed points under the action of $\mathrm{Stab}_{H_7}(\ell) \simeq \mathbf{Z}_7$ and then an orbit of fourteen points on C. Noticing all the components have the same stabilizer (the only group morphism from $\mathbf{Z}_7$ to the symmetric group $\mathfrak{S}_6$ is constant) and considering the symmetries of G_7 it is easy to prove that these fourteen points coincide two by two, implying C is a heptagon.

If the singular locus C has degree 14, then the reduced structure of it has degree 7 or 14. Only the latter is a problem. The normalization of the surface has sectional genus (-6) so it consists of at least seven components. They have the same degree *i.e.* 1 or 2, so their number must be 7 and their degree must be 2. In particular, either the surface A is contained in one orbit of seven $\mathbf{P}_3$'s under G_7 or the reduced structure of A consists of seven planes. In both cases, it is a simple matter to conclude for the only orbits of seven $\mathbf{P}_3$'s under G_7 are listed in the subsection 3.2 (consider for instance their possible instersections with the forty-nine $\mathbf{P}_3^+$) so the surface A appears already in the subsection 3.2 and the proposition is true for such surfaces.

The last possible case is when C has degree 21, but then the surface A has 14 components (its normalization has sectional genus (-13)) so C splits and, as $\gcd(49, 21) = 7$, contains three G_7-invariant curves of degree 7 so we are back to the first case. □

End of the proof of the theorem.
Let A be a generalized abelian surface, preimage of a six secant plane of $\kappa(\mathbf{P}_2^+)$ by κ, *i.e.* $\zeta_A = (1,1,1,1,1,1)$. Since $A \cap \mathscr{K}'_4 = \emptyset$ we may divide in two cases; A intersects $\mathscr{H}_6$ but not $\mathscr{K}'_4$ in $\mathbf{P}_2^+$, and A intersects neither $\mathscr{H}_6$ not $\mathscr{K}'_4$.

i. Assume A intersects $\mathscr{H}_6$ in $\mathbf{P}_2^+$, then A is a *smooth* plane curve fibration and has a trisecant line in $\mathbf{P}_2^+$: see construction in [11];

ii. Assume $A \cap \mathscr{H}_6 = A \cap \mathscr{K}'_4 = \emptyset$. Otherwise A is a translation scroll or a plane curve fibration. By the previous lemma, A is irreducible with isolated singularities. Let us choose $a \in \mathbf{P}_2^+ \cap A$ and consider (identifying $\mathbf{P}_2^+$ with $\mathbf{P}W_3$) the Calabi Yau threefold Y_a preimage of the cone Γ_a by κ. We know that $Y_a = \bigcup_{t \in \mathcal{C}_a} A_t$. Y_a has a quadratic singularity at a, and the general surface A_t is smooth at a, so after a small resolution A_t will be Cartier there. In the following we will ignore this distinction between Y_a and its small resolution of a and its G_7 translates. We have

- $\omega_A = \mathscr{O}_A$. Indeed, the general A_t is smooth (since it is for $t \in \mathscr{H}_6$, notice also that $\bigcup_{t \in \mathcal{C}_a}\{A_t\}$ contains 8 translation scrolls and 4 elliptic curve fibrations) with normal bundle $\mathscr{O}_A(A) = \mathscr{O}_A$ so $\omega_A = \mathscr{O}_A(K_{Y_a}) = \mathscr{O}_A$;

- A is a Cartier divisor on the Calabi Yau threefold Y_a: the threefold Y_a is mapped by definition to a quadric Γ_a of rank 4 (this follows easily considering a is not on $\mathscr{H}_6$) so A is a Cartier divisor on Y_A providing $\bigcap_{t\in\mathcal{C}_a} = G_7 \cdot a$ *i.e.* if the cardinality of $\bigcap_{t\in\mathcal{C}_a}$ is 49.
  Let $B_a = \bigcap_{t\in\mathcal{C}_a}$. If B_a contains a curve D then it has degree at least 14. On the other hand it must lie on a the translation scrolls of the pencil A_t. Let T be such a scroll and $\widetilde{T}$ its desingularization. Then $\mathrm{Pic}(\widetilde{T}) =< E_0, F >$, where E_0 is a section with $E_0^2 = 0$ and F is a fibre. The preimage of the singular curve on T is the union of two sections E_0 and E_0'. Now, B_a cannot intersect the singular curve on T, for degree reasons: A_t is cut out by cubics, while the intersection would have cardinality at least 49. Therefore $D \cdot E_0 = 0$. But then $D = \alpha \cdot E_0 + \beta \cdot E_0'$, which is again a contradiction. So B_a is finite.
  First $B_a \cap \mathbf{P}_3^- = \emptyset$: by remark **3.5**, the points of $\mathbf{P}_3^-$ contained in a pencil of abelian surfaces are perfectly idenfified — as well as the corresponding abelian surfaces: these are either translation scrolls or plane curve fibrations —. Assume B_a has cardinality greater than 49, then because of the previous argument it has cardinality greater than $147 = 49 + 2 \times 49$ where 2×49 is the cardinality of G_7 (acting on $\mathbf{P}V_0$). But B_a is contained in the base locus B of the map κ so $\kappa|_{A\ smooth}$ has at least $147 + 10 \times 49$ (147 contained in B_a and the other ones coming from the intersection of A with the 49 $\mathbf{P}_3^-$'s) base points, another contradiction.

$\square$

4 Appendix

4.1 Some pictures

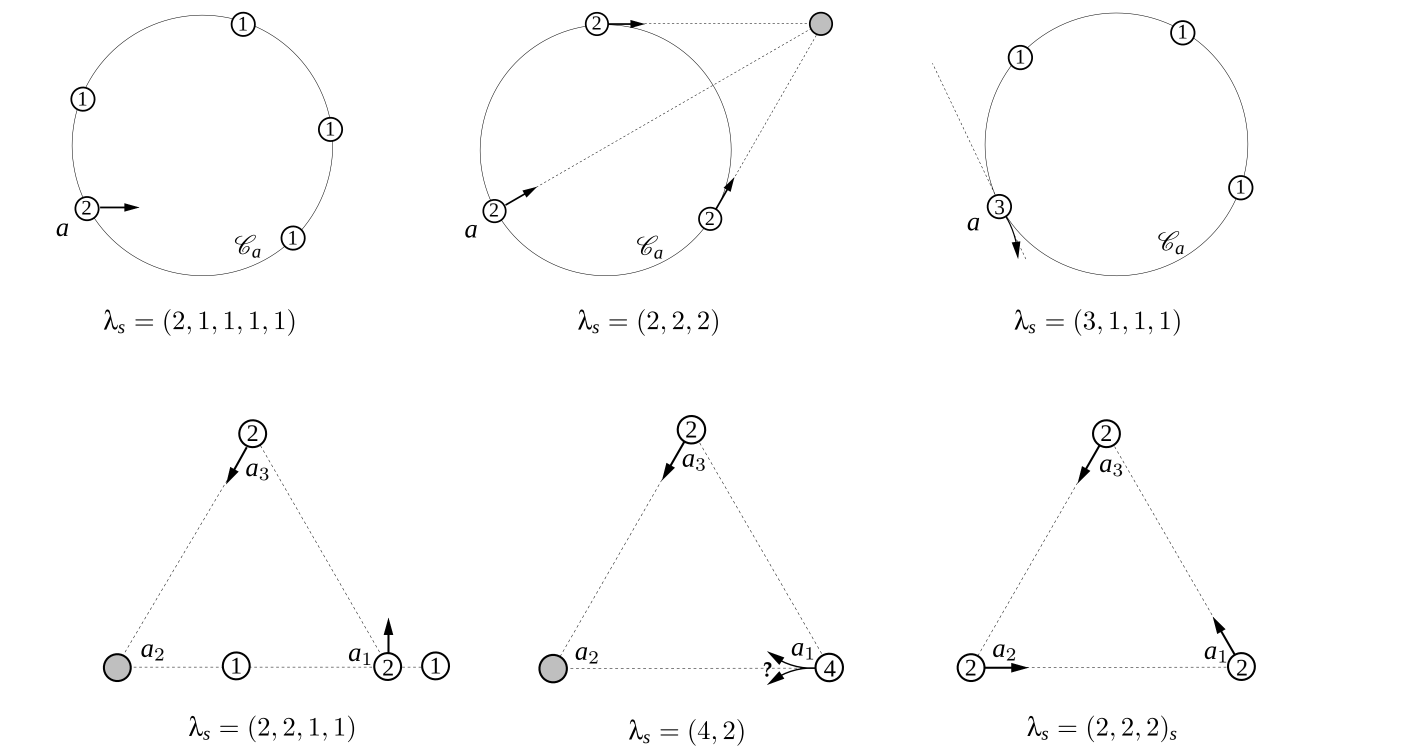

FIG. 1 — *Possible configurations of ζ_s when $s \in \Delta$, where each arrow gives the (first) direction along which the point is doubled (oriented in a purely decorative way).*

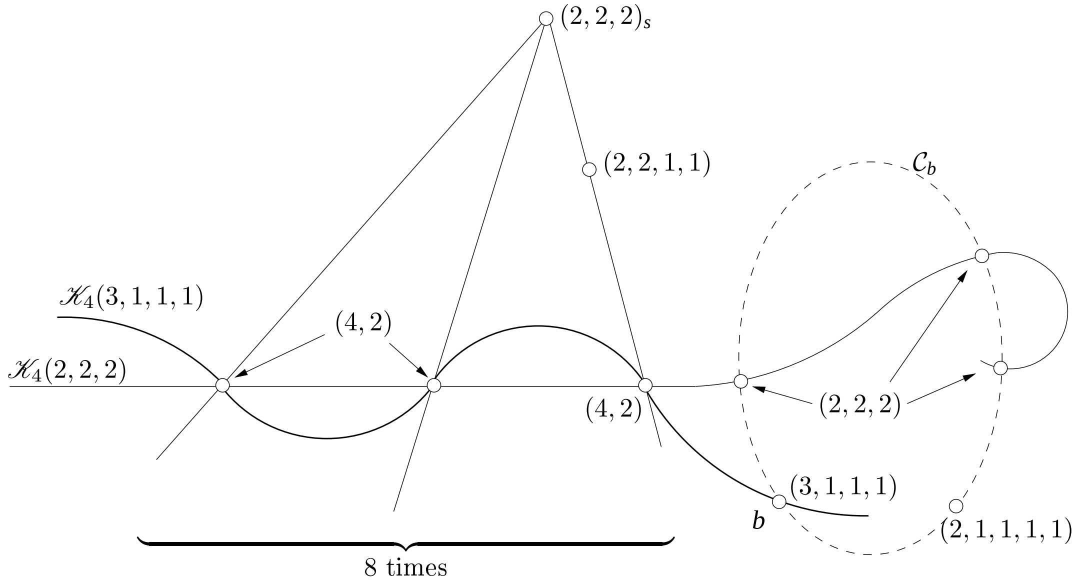

FIG. 2 — *Stratification of the surface* Δ.

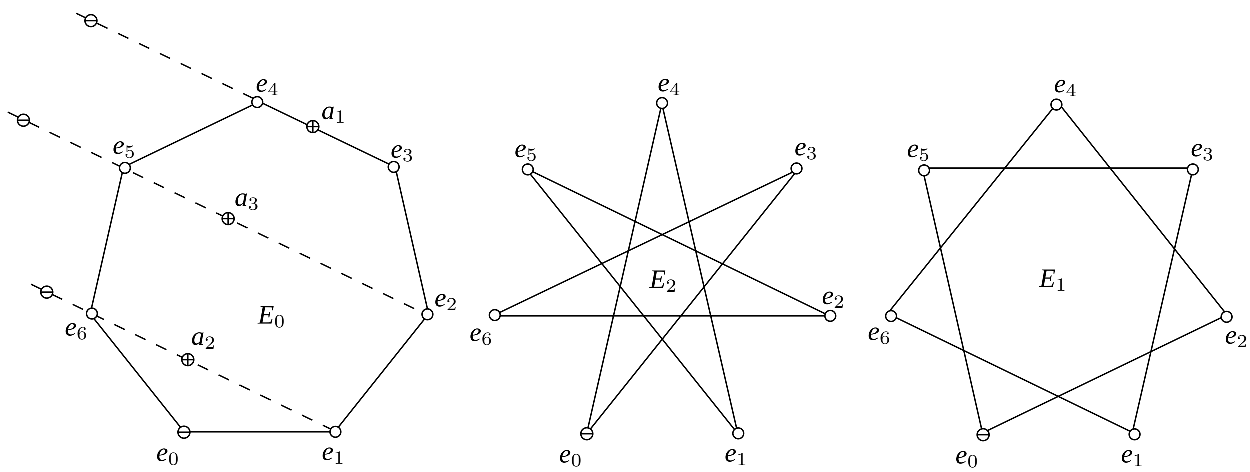

FIG. 3 — *A triplet of degenerate elliptic curves.*

4.2 Some questions

Let A be a smooth abelian surface embedded in $\mathbf{P}V_0$. As already noticed in proposition **1.6** we have a factorization

$$A^\flat \xrightarrow{49:1} A^{\vee\flat} \xrightarrow{2:1} K^\flat_{A^\vee} \xrightarrow{2:1} \kappa(A)\ .$$

What is the ramification locus of the last map? Of course, the map to the right needs to define a $K3$ surface so it is certainly a sextic curve (of genus 4), but it doesn't explain how to recover it (using representation theory for instance). A good understanding of this point should enable one to get a reconstruction method, as in the $(1,5)$ case, of the abelian surface $A^\vee$. Note that this ramification locus admits the six points $\kappa(A)\cap\kappa(\mathbf{P}_2^+)$ as double points. Next the surface A intersects $\mathbf{P}_3^-$ in ten points, so projecting A from $\mathbf{P}_3^-$ we get maps

$$A^\flat \xrightarrow{2:1} K^\flat_A \xrightarrow{2:1} \mathbf{P}_2^+\ .$$

The last map is ramified along a sextic curve with the six points $A \cap \mathbf{P}_2^+$ as double points. Hence we have two maps $K_A^\flat \xrightarrow{2:1} \mathbf{P}_2^+$ and $K_A^\flat \xrightarrow{2:1} \kappa(A^\vee)$.

Is it possible to find an (in fact 168*) indentification(s) between* $\mathbf{P}_2^+$ *and* $\kappa(A^\vee)$ *such that the two maps coincide?* The answer is positive in the $(1,5)$ case and allows one to identify the moduli space of $(1,5)$ polarized abelian surfaces (without level structure) up to duality. Note that one can easily show that the two sets of six points $A \cap \mathbf{P}_2^+$ and $\kappa(A) \cap \kappa(\mathbf{P}_2^+)$ are associated in Coble's sense. This phenomenom is in fact true for any Fano threefold V_{22}: given a six secant plane to $\mathscr{V}'_{2,9}$ (Veronese surface isomorphic to $\mathbf{P}W$), it intersects it along six points associated to the corresponding set in $\mathbf{P}W$.

It is possible to show that the Fano threefold $VSP(\mathscr{K}_4, 6)$ is 'stable' by association of points *i.e.* that there exists a (dual) Klein curve C in the plane $\kappa(A)$ such that $\kappa(A) \cap \kappa(\mathbf{P}_2^+)$ is a point of the corresponding variety $VSP(C, 6)$, isomorphic, up to $\mathbf{P}\mathrm{SL}(2, \mathbf{F}_7)$, to $VSP(\mathscr{K}_4, 6)$. Is the quotient the moduli space of $(1,7)$ polarized abelian surfaces (without level structure) up to duality?

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